

Escape from a circle and Riemann hypotheses

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Abstract. We consider open circular billiards with one and with two holes. The exact formulas for escape are obtained which involve the Riemann zeta function and Dirichlet L functions. It is shown that the problem of finding the exact asymptotics in the small hole limit for escape in some of these billiards is equivalent to the Riemann hypothesis.

1. Introduction

The theory of open dynamical systems where orbits may disappear upon reaching some region in phase space just recently started to attract the attention of mathematicians. A natural reason for this is because it is much harder in general to study open systems in comparison with closed dynamical systems. The theory of closed dynamical systems is rather well developed. Therefore these relatively few mathematical studies of open dynamical systems are naturally based on ideas and techniques developed in dealing with closed systems. However, the studies of open systems may also bring new insights to the understanding of dynamics of closed systems as well.

One of possible paths to explore in this direction was recently suggested in Ref. [1]. Let us consider open systems with several “holes” (regions where orbits disappear upon hitting them) and compare behaviors of such “many-holes” systems with the corresponding single-hole systems. The idea is that such comparison may shed a light on understanding dynamics of a closed system one gets by “patching” all the holes in the open systems. (This approach could be of interest for geometric theory of dynamical systems by studying e.g., geodesic flows on manifolds with “holes.”)

This idea has also a lot of potential applications for the real world “physical” systems [1]. Indeed, in experimental studies researchers perform measurements outside a region of interest (“container”) by e.g., measuring fluxes out of the container [2, 3, 4]. Actually our approach arose from the claim made by one of the authors, who was inspired by experiments with the optical billiards, that a comparison of escape rates through one and through two holes may shed some light on the dynamics of the corresponding closed systems. This approach may even have some potential industrial applications related, e.g., to the optimal placement of the holes in order to maximize (or minimize) corresponding fluxes.

One of the most natural classes of the open systems to study is formed by the open billiards. The first application of the one-many holes interplay idea has already led to a very surprising and, in a sense, remarkable result. Seemingly the simplest problem of this type, namely the comparison of dynamics of the circular billiards with one and with two opposite holes, is equivalent to the Riemann hypothesis (RH) [1]. Formally, this

result opens up a possibility to verify RH in real physical experiments. (It does not seem however that it is going to be very fruitful and, especially, practical approach because the modern computers provide more efficient tools for that in numerical experiments.) On the other hand, this result demonstrates that indeed the studies of open systems (and particularly of open billiards) may bring about some interesting and unexpected advances in traditional areas of research.

The purpose of this paper is to provide the proofs of the results announced in [1].

2. Definitions and notations

Consider a billiard on the unit disk D i.e., a dynamical system generated by the motion of a point particle with a constant speed within D with elastic collisions (angle of incidence equals angle of reflection) from its boundary. Without any loss of generality we assume that the particle's speed is identically one, and therefore its velocity is completely defined by an angle ϑ it makes with the horizontal direction, $-\pi < \vartheta \leq \pi$. Billiards are Hamiltonian systems. Therefore the Liouvillean measure (the phase volume) in the phase space \mathcal{M} is preserved under the dynamics (a billiard flow) $\{S^t\}$, $-\infty < t < \infty$.

Let $M = \{(\beta, \psi) : -\pi < \beta \leq \pi, -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\}$, where $\beta \in \partial D$. The billiard flow in D induces the billiard map $T : M \rightarrow M$ defined as

$$T(\beta, \psi) = (\beta + \pi - 2\psi, \psi) \quad (1)$$

where ψ is the angle between the outward trajectory and the inner normal at $\beta \in \partial D$, and all the angles in (1) are taken modulo 2π . The natural projection $M \rightarrow \partial D$ we denote by proj . Thus $\text{proj}(\beta, \psi) = \beta$, where $(\beta, \psi) \in M$.

It is well known that orbits of the billiard in a circle are either periodic with a period q (if $\psi = \frac{\pi}{2} - \frac{p}{q}\pi$, where p and q are co-prime integers and $p < q$) or everywhere dense in ∂D (if ψ is incommensurable with π). The billiard map T preserves the measure

$$d\mu = \frac{\cos \psi}{4\pi} d\psi d\beta \quad (2)$$

which is the projection of the Liouvillean measure in the phase space \mathcal{M} of the billiard flow onto M .

Suppose that two (possibly overlapping) holes $H_1 = \{\beta, 0 < \beta < \Delta\}$ and $H_2 = \{\beta : \theta < \beta < \theta + \Delta\}$, $0 \leq \theta < 2\pi$, $\Delta > 0$, are placed at the boundary ∂D .

Consider now a new dynamical system, an open billiard in ∂D with holes H_1 and H_2 . In this open billiard any orbit (β_0, ψ_0) moves under the billiard map (1) until it hits one of the holes H_1 and H_2 . When the orbit hits $H_1 \cup H_2$ it “disappears” (escapes).

Obviously, almost all (with respect to the measure μ) orbits will eventually escape. The only orbits that never escape are such periodic orbits that never hit $H_1 \cup H_2$. Denote $\widehat{H}_i = \{(\beta, \psi) : \beta \in H_i\}$, $i = 1, 2$. Thus $\text{proj } \widehat{H}_i = H_i$, $i = 1, 2$. By $\text{dist}(\beta_1, \beta_2)$, $\beta_i \in \partial D$, $i = 1, 2$, we will mean the length of the shortest arc between β_1 and β_2 .

Let $N(\beta_0, \psi_0)$, $(\beta_0, \psi_0) \in M$ be a (minimal) number of reflections from the boundary after which the orbit $T^n(\beta_0, \psi_0) = (\beta_n, \psi_n)$, $n = 1, 2, \dots$ escapes from the circle. (If the orbit of (β_0, ψ_0) never escapes we set $N(\beta_0, \psi_0) = \infty$.) Therefore the orbit (β_0, ψ_0) escapes from the circle in a (real, continuous) time $\tau(\beta_0, \psi_0) = 2 \cos \psi_0 N(\beta_0, \psi_0)$.

3. Structure of the set of orbits not escaping in time t

Clearly, the only orbits that never escape are those periodic orbits with periods $q < \frac{2\pi}{\Delta}$ which never hit the holes $H_1 \cup H_2$.

All orbits of a rotation of the circle at any irrational (with respect to π) angle are everywhere dense (on the circle). Therefore all orbits of irrational rotations of the circle eventually escape regardless of the size $\Delta > 0$ of the holes.

We will always assume in what follows that $\Delta < \frac{\pi}{2}$. Then there exists a periodic orbit of period two which never escapes.

The following statement obviously holds.

Lemma 1 *If $\Delta < \frac{\pi}{n}$, $n \geq 2$, then there exists a periodic orbit of period n which never escapes.*

Lemma 2 *If $\Delta < \frac{\pi}{n}$, $n \geq 2$, then for any $t > 0$ there exists a nonperiodic orbit which does not escape till time t .*

Proof. Let $\frac{\pi}{n} - \Delta = \delta > 0$. By Lemma 1 there exists a periodic orbit $(\beta_i, \hat{\psi})$, $i = 1, 2, \dots, n$, $T(\beta_i, \hat{\psi}) = (\beta_{i+1}, \hat{\psi})$ if $1 \leq i \leq n-1$, $T(\beta_n, \hat{\psi}) = (\beta_1, \hat{\psi})$, such that $\min_{1 \leq j \leq n} \text{dist}(\beta_j, \{H_1 \cup H_2\}) = \frac{\delta}{2}$. Therefore for any $t > 0$ and for any j , $1 \leq j \leq n$, there exists such $\alpha_{j,t} = \alpha_{j,t}(\delta)$ that orbits of all points (β_j, ψ) with $|\psi - \hat{\psi}| < \alpha_{j,t}$ do not escape from the circle till time t .

Clearly, smaller the difference $|\psi - \hat{\psi}|$ is, longer the corresponding orbit will not escape.

Denote by \mathcal{N}_t the set of all orbits that do not escape till the time t . We will show that for sufficiently large t , the set \mathcal{N}_t can be decomposed into the union of nonintersecting neighborhoods of never escaping periodic orbits.

Lemma 3 *Let $x' = (\beta, \psi')$ and $x'' = (\beta, \psi'')$ be the points of two never escaping periodic orbits with periods n' and n'' respectively. Then x' and x'' belong to different connected components of the sets \mathcal{N}_t if $t > 2(n' + n'' - 1) \max(\cos \psi', \cos \psi'')$.*

Proof. Suppose $n'' > n'$ and $n'' \neq kn'$, where $k > 0$ is an integer. Consider the set $A_{\beta, \psi', \psi''} = \{(\beta, \psi) : \psi' \leq \psi \leq \psi''\}$. (We assumed here that $\psi'' > \psi'$. But the same argument can be applied if $\psi'' < \psi'$.) Clearly $\text{proj}(TA_{\beta, \psi', \psi''})$ is the arc of ∂D between the points $\text{proj}(Tx')$ and $\text{proj}(Tx'')$. Analogously $\text{proj}(T^m A_{\beta, \psi', \psi''})$, $1 \leq m \leq n'$, is the arc between $\text{proj}(T^m x')$ and $\text{proj}(T^m x'')$.

Now we show that $\text{proj}\left(\bigcup_{m=1}^{n'+n''-1} T^m A_{\beta, \psi', \psi''}\right) = \partial D$. Recall that x' and x'' are never escaping periodic orbits. Observe that the circle ∂D is divided by the points $\text{proj}(T^m x')$, $0 \leq m \leq n' - 1$, and $\text{proj}(T^m x'')$, $0 \leq m \leq n'' - 1$ into at most $n' + n'' - 1$ arcs. We will call these arcs minimal arcs. Clearly each arc between $\text{proj}(T^m x')$ and $\text{proj}(T^m x'')$, covers at least one minimal arc that was not covered by the arcs between $\text{proj}(T^{m-1} x')$ and $\text{proj}(T^{m-1} x'')$. Indeed, otherwise the corresponding preimage of this arc also covered only minimal arcs already covered by the previous iterations of T . Continuing this argument we will come back to the point $\beta \in \partial D$ and to a contradiction. Therefore, not more than by $(n' + n'' - 1)$ iterations all the circle ∂D will be covered. Hence, within $(n' + n'' - 1)$ iterates of the billiard map T the projections on ∂D of the corresponding images of the set $A_{\beta, \psi', \psi''}$ will completely cover the hole H_1 and the hole H_2 . Thus after that time the iterates of x' and x'' will belong to different connected components of the sets \mathcal{N}_t . The case $n'' = kn'$ is even simpler to consider and the analysis goes along the same lines.

We will show now that any connected component of the set \mathcal{N}_t contains some interval $I_{\psi_{\frac{m}{n}, \beta_1, \beta_2}} = \{(\beta, \psi) : \psi = \frac{\pi}{2} - \frac{m}{n}\pi, \beta_1 \leq \beta \leq \beta_2\}$. Observe that the orbit of the point $(\beta, \frac{\pi}{2} - \frac{m}{n}\pi)$ has period n and the β values are equally spaced at intervals of $\frac{2\pi}{n}$.

The next statement assures that any connected component of the set of nonescaping orbits \mathcal{N}_t for any t contains never escaping periodic orbits. Let (a, b) denote the gcd (a, b) .

Lemma 4 *All nonempty connected components of the set \mathcal{N}_t for all $t > 0$ contain a point of a never escaping periodic orbit.*

Proof. Let m_i/n_i , $(m_i, n_i) = 1$, $i = 1, 2$ be consecutive fractions among all m/n , $(m, n) = 1$, $\frac{2\pi}{n} > \Delta$, i.e. Farey numbers. Assume at first that for some β periodic orbits of both points $x_1 = (\beta, \frac{\pi}{2} - \frac{m_1}{n_1}\pi)$, $x_2 = (\beta, \frac{\pi}{2} - \frac{m_2}{n_2}\pi)$ never escape.

Consider the sets

$$A^{(1)} = A_{\beta, \frac{\pi}{2} - \frac{m_1}{n_1}\pi, \frac{\pi}{2} - \frac{m_1}{n_1}\pi + \frac{\pi}{n_1(n_1+n_2)}}^{(1)} \quad \text{and} \\ A^{(2)} = A_{\beta, \frac{\pi}{2} - \frac{m_2}{n_2}\pi - \frac{\pi}{n_2(n_1+n_2)}, \frac{\pi}{2} - \frac{m_2}{n_2}\pi}^{(2)}$$

Because $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$ are the consecutive Farey numbers (with denominators not exceeding $[\frac{2\pi}{\Delta}]$)

$$A_{\beta, \frac{\pi}{2} - \frac{m_1}{n_1}\pi, \frac{\pi}{2} - \frac{m_2}{n_2}\pi} = A^{(1)} \cup A^{(2)},$$

where $[a]$ denotes the integer part of the number a .

Consider $\text{proj}(T^{n_i}A^{(i)})$, $i = 1, 2$. It is easy to calculate that the length of each of these two arcs of ∂D equals $2\pi/(n_1 + n_2)$. The well known property of Farey numbers ensures that $(n_1 + n_2) > [\frac{2\pi}{\Delta}] + 1$. Therefore the distance between the projection β of any point $(\beta, \psi) \in A^{(i)}$, $i = 1, 2$, and the projection $\text{proj}(T^{n_i}(\beta, \psi))$ does not exceed $\frac{2\pi}{n_1+n_2} < \Delta$.

The last inequality ensures that all images of the point $(\beta, \psi) \in A^{(1)} \cup A^{(2)}$ belong (before escaping) to the same connected component of the sets \mathcal{N}_t to which belong at least one point of the periodic orbit $(\beta, \frac{\pi}{2} - \frac{m_1}{n_1}\pi)$ or of $(\beta, \frac{\pi}{2} - \frac{m_2}{n_2}\pi)$ or of each of these periodic orbits (if t is sufficiently small).

Let now assume that one or both of the points x_i , $i = 1, 2$ do escape. Then for some $k > 0$ $\text{proj}(T^k x_1)$ belongs to $H_1 \cup H_2$. (The consideration is absolutely analogous if the orbit of x_2 escapes at some bounce off the boundary of the disk D .) Consider all the points of the set $T^k A^{(1)}$ that have not escaped to this moment. Among these points must be one with projection at the end point of the hole H_i , where $T^k x_1$ escaped.

Consider now the point x_3 that goes from this point under the angle $\frac{\pi}{2} - \frac{m_1}{n_1}\pi$. Obviously this point is periodic and the corresponding periodic orbit will never escape.

Consider all the points of the set $T^k A^{(1)}$ that have not escaped to this moment. Clearly, the length of the projection of this set onto the boundary of D does not exceed $\pi/(n_1 + n_2) < \Delta/2$. Therefore all periodic orbits that go from the points of this set under the angle $\pi/2 - \pi m_1/n_1$ will never escape.

On the other hand from the above argument in the proof it follows that all the points of the images of the set $A^{(1)}$ will belong (until the escape) to the same connected component of the set \mathcal{N}_t as the images of the never escaping periodic point $T^{-k}x_3$. This completes the proof of Lemma 4.

Lemmas 3 and 4 imply that the following statement holds.

Theorem 1 *Let $t > 4[\frac{2\pi}{\Delta}]$. Then every connected component B_i , $i = 1, 2, \dots, m$ $m = m(\Delta)$ of the set \mathcal{N}_t of orbits never escaping till time t contains a unique segment $I_i = \{(\beta, \psi), \quad \beta_{i,1} < \beta < \beta_{i,2}\}$ consisting of never escaping periodic orbits.*

4. Probability of not escaping till time t

We now compute the measure $\mu(\mathcal{N}_t)$ of the set of all orbits that do not escape till time t . Lemmas 1–4 imply that $\mu(\mathcal{N}_t) > 0$ for any $t < \infty$.

Denote $\psi_{m,n} = \frac{\pi}{2} - \frac{m}{n}\pi$, where $m < n$, $(m, n) = 1$, $n < [\frac{2\pi}{\Delta}]$. Clearly $\mathcal{N}_t \subset M \setminus \bigcup_{k=0}^{n-1} (T^{-k}(\hat{H}_1 \cup \hat{H}_2))$ if $t > 2[\frac{2\pi}{\Delta}]$. In what follows we will always assume that $t > \frac{8\pi}{\Delta}$.

Let $(\beta, \psi) \in \mathcal{N}_t$. Then in view of Lemma 4 and Theorem 1 the coordinate ψ can be uniquely represented as $\psi = \psi_{m,n} + \eta$, where $|\eta| < \Delta/2$. It is easy to see that the orbit of the point (β, ψ) escapes not later than at the time t if there exists such integer k , $0 \leq k \leq \left\lfloor \frac{t}{2 \sin(\frac{m}{n}\pi + \eta)} \right\rfloor$ that

$$\text{proj } T^k(\beta, \psi) \in H_1 \cup H_2 \quad (3)$$

Denote by $T_{m,n}$ rotation of the circle ∂D on the angle $\psi_{m,n}$. Lemma 4 and Theorem 1 ensure that every connected component of the set \mathcal{N}_t can be uniquely represented as $\mathcal{N}_{\psi_{m,n},j}^t$ where $m < n$, $(m, n) = 1$, $n < [\frac{2\pi}{\Delta}]$ and $0 \leq j \leq 2n - 1$ if $H_2 \cap (H_1 \cup T_{m,n}H_1 \cup T_{m,n}^2H_1 \cup \dots \cup T_{m,n}^{n-1}H_1) = \emptyset$ or, otherwise, $0 \leq j \leq n - 1$.

Let $\theta' = \theta(\bmod \frac{2\pi}{n})$. If $\theta' < \Delta$ then the set $\partial D \setminus \bigcup_{k=0}^{n-1} T_{m,n}^k(H_1 \cup H_2)$ consists of n arcs of the length $\frac{2\pi}{n} - \theta' - \Delta$, otherwise it consists of $2n$ arcs, n of which are of the length $\frac{2\pi}{n} - \theta' - \Delta$ and another n arcs are of the length $\theta' - \Delta$. We will call these arcs complements to a hole's orbit.

Thus one can write

$$\mathcal{N}_t = \bigcup_{\substack{m < n \\ (m,n)=1 \\ n < \lceil \frac{2\pi}{\Delta} \rceil}} \bigcup_j \mathcal{N}_{\psi_{m,n,j}}^t \quad (4)$$

where $0 \leq j \leq 2n-1$ or $0 \leq j \leq n-1$. Consider now all connected components $\mathcal{N}_{\psi_{m,n,j}}^t$, $0 \leq j \leq 2n-1$ (the case when $0 \leq j \leq n-1$ can be treated analogously and, in fact, is slightly simpler). According to Lemma 4 and Theorem 1 all $\mathcal{N}_{\psi_{m,n,j}}^t$ are closed sets for any m, n, j .

We will call a connected component $\mathcal{N}_{\psi_{m,n,j}}^t$ a basic component if $\text{proj}(\mathcal{N}_{\psi_{m,n,j}}^t)$ is adjacent to a hole H_1 or H_2 . (Observe that in the case when $0 \leq j \leq 2n-1$ there are four basic components for a fixed m, n while if $0 \leq j \leq n-1$ then there are two basic components.) In each basic component we will single out a closed subset which will be called a basic set in what follows (instead of its longer name a basic subset of a basic component).

Definition 1 A basic subset $\widehat{\mathcal{N}}_{\psi_{m,n,j}}^t \subset \mathcal{N}_{\psi_{m,n,j}}^t$ consists of all points $(\beta, \psi) \in \mathcal{N}_{\psi_{m,n,j}}^t$ such that

$$\text{dist}(\beta, H) \leq \text{dist}(\text{proj}(T^n(\beta, \psi)), H)$$

where H is a hole adjacent to $\text{proj}(\mathcal{N}_{\psi_{m,n,j}}^t)$, i.e., H_1 or H_2 .

Observe that the above inequality is well defined because $\text{dist}(\beta, \text{proj}(T^n(\beta, \psi))) < \Delta$. It is easy to see that each basic component $\mathcal{N}_{\psi_{m,n,j}}^t$ contains one and only one basic subset. Therefore for any pair of positive integers $m < n$, $(m, n) = 1$, $\lceil \frac{2\pi}{n} \rceil > \Delta$ there exist four (in the case we consider, otherwise two) basic subsets $\widehat{\mathcal{N}}_{m,n}^{t,i}$, $i = 1, 2, 3, 4$.

A crucial fact is that for any point $(\beta, \psi) \in \mathcal{N}_{\psi_{m,n}}^t$ there exist one and only one k , $0 \leq k < n$, and one and only one i , $i = 1, 2, 3, 4$, such that $T^k(\beta, \psi) \in \widehat{\mathcal{N}}_{m,n}^{t,i}$. Indeed, it follows from the definition of the basic set, Lemma 3 and the obvious relation that $\text{proj}(T^k(\beta, \psi)) \notin H_1 \cup H_2$ for any $0 \leq k < n$.

Certainly $\text{proj}(\widehat{\mathcal{N}}_{m,n}^{t,i}) \cap (\bigcup_{k=0}^{n-1} T_{m,n}^k(H_1 \cup H_2)) = \emptyset$ for $i = 1, 2, 3, 4$. Therefore projections of two ($i = 1, 2$) basic sets have the length $\frac{2\pi}{n} - \theta' - \Delta$, and of two others ($i = 3, 4$) $\theta' - \Delta$. (Recall that we consider only the case $0 \leq j \leq 2n - 1$.)

It follows from the definition of the basic sets that if $(\beta, \psi) \in \widehat{\mathcal{N}}_{m,n}^{t,1}$ then $\psi \geq \psi_{m,n}$ while if $(\beta, \psi) \in \widehat{\mathcal{N}}_{m,n}^{t,2}$ then $\psi \leq \psi_{m,n}$ (or vice versa). The same statement is true for the basic sets $\widehat{\mathcal{N}}_{m,n}^{t,3}$ and $\widehat{\mathcal{N}}_{m,n}^{t,4}$, i.e., for $i = 3$ and $i = 4$.

Consider now the sets $\widetilde{\mathcal{N}}_{m,n,k}^t, \widetilde{\mathcal{N}}_{m,n,k}^t \subset \mathcal{N}_{\psi_{m,n}}^t$, $k = 0, 1, \dots, n-1$, where

$$\widetilde{\mathcal{N}}_{m,n,k}^t = \left\{ (\beta, \psi) \in \mathcal{N}_{\psi_{m,n}}^t : T^k(\beta, \psi) \in \widehat{\mathcal{N}}_{m,n}^{t,1} \cup \widehat{\mathcal{N}}_{m,n}^{t,2} \right\}$$

and

$$\widetilde{\mathcal{N}}_{m,n,k}^t = \left\{ (\beta, \psi) \in \mathcal{N}_{\psi_{m,n}}^t : T^k(\beta, \psi) \in \widehat{\mathcal{N}}_{m,n}^{t,3} \cup \widehat{\mathcal{N}}_{m,n}^{t,4} \right\}.$$

It is easy to see that the sets $\widetilde{\mathcal{N}}_{m,n,k}^t$ and $\widetilde{\mathcal{N}}_{m,n,k}^t$ are disjoint. Moreover, $\widetilde{\mathcal{N}}_{m,n,k_1}^t \cap \widetilde{\mathcal{N}}_{m,n,k_2}^t = \emptyset$ ($\widetilde{\mathcal{N}}_{m,n,k_1}^t \cap \widetilde{\mathcal{N}}_{m,n,k_2}^t = \emptyset$) if $k_1 \neq k_2$. Therefore we can write

$$\mathcal{N}_t = \bigcup_{\substack{m < n \\ (m,n)=1 \\ n < [\frac{2\pi}{\Delta}]}} \bigcup_{k=0}^{n-1} \left(\widetilde{\mathcal{N}}_{m,n,k}^t \cup \widetilde{\mathcal{N}}_{m,n,k}^t \right) \quad (5)$$

It follows from (5) and (2) that

$$\begin{aligned} \mu(\mathcal{N}_t) = & \sum_{\substack{m < n \\ (m,n)=1 \\ n < [\frac{2\pi}{\Delta}]}} \sum_{k=0}^{n-1} \left(\int_0^{\frac{2\pi}{n} - \theta' - \Delta} d\beta \int_{-\eta_{m,n}^-(\beta,k)}^{\eta_{m,n}^+(\beta,k)} \sin(\psi_{m,n} + \eta) d\eta \right. \\ & \left. + \int_0^{\theta' - \Delta} d\beta \int_{-\eta_{m,n}^-(\beta,k)}^{\eta_{m,n}^+(\beta,k)} \sin(\psi_{m,n} + \eta) d\eta \right) \end{aligned} \quad (6)$$

where the coordinates β on $\text{proj}(\widehat{\mathcal{N}}_{m,n}^{t,1})$ and on $\text{proj}(\widehat{\mathcal{N}}_{m,n}^{t,2})$ (or on $\text{proj}(\widehat{\mathcal{N}}_{m,n}^{t,3})$ and on $\text{proj}(\widehat{\mathcal{N}}_{m,n}^{t,4})$) are naturally identified. The orbit $(\beta, \psi_{m,n} + \eta)$ escapes not later than at the time t if there exists such integer r , $0 \leq r \leq [\frac{t}{2 \sin(\frac{t}{n} \pi - \eta)}]$ that $\text{proj } T^r(\beta, \psi) \in H_1 \cup H_2$.

Denote $\mathcal{N}_{\beta_0, \psi_{m,n}, \ell}^t = \mathcal{N}_{\psi_{m,n}, \ell}^t \cap \{(\beta, \psi) : \beta = \beta_0\}$, $0 \leq \ell < n$. Then, in view of Lemma 4 and Theorem 1

$$\mathcal{N}_{\beta, \psi_{m,n}, \ell}^t = \{(\beta, \psi) : \psi_{m,n} + \eta_{m,n}^- \leq \psi \leq \psi_{m,n} + \eta_{m,n}^+\},$$

where $\eta_{m,n}^- = \eta_{m,n}^-(\beta, \ell)$, $\eta_{m,n}^+ = \eta_{m,n}^+(\beta, \ell)$, $0 \leq \ell < n$, but we will often drop the dependence on β and ℓ to simplify notations.

Clearly $t/2 \cos(\psi_{m,n} + \eta_{m,n}^-)$ and $t/2 \cos(\psi_{m,n} + \eta_{m,n}^+)$ are both integers. Moreover, for any $\eta > \eta_{m,n}^+$ ($\eta > \eta_{m,n}^-$) the escape occurs before the time t . Therefore, either $\text{proj } T^{[t/2 \cos(\psi_{m,n} + \eta_{m,n}^-)]}(\beta, \psi_{m,n} + \eta_{m,n}^-) = (0, \psi_{m,n} + \eta_{m,n}^-)$ or $\text{proj } T^{[t/2 \cos(\psi_{m,n} + \eta_{m,n}^-)]}(\beta, \psi_{m,n} + \eta_{m,n}^-) = (\theta, \psi_{m,n} + \eta_{m,n}^-)$.

Correspondingly, either

$\text{proj } T^{[t/2 \cos(\psi_{m,n} + \eta_{m,n}^+)]}(\beta, \psi_{m,n} + \eta_{m,n}^+) = (\Delta, \psi_{m,n} + \eta_{m,n}^+)$ or $\text{proj } T^{[t/2 \cos(\psi_{m,n} + \eta_{m,n}^+)]}(\beta, \psi_{m,n} + \eta_{m,n}^+) = (\Delta + \theta, \psi_{m,n} + \eta_{m,n}^+)$.

Denote by $\rho_{\beta,m,n}^+$ ($\rho_{\beta,m,n}^-$) the distances (the lengths of the arcs) from β to the ends (of the closest to β) connected components of $\text{proj}(\mathcal{N}_{\psi_{m,n}}^t)$. Then, according to (1), we have

$$\begin{aligned}\rho_{\beta,m,n}^+ &= (2\eta_{m,n}^+(\beta, \ell))r_\ell(\beta, m, n) \\ \rho_{\beta,m,n}^- &= (2\eta_{m,n}^-(\beta, \ell))r_\ell(\beta, m, n)\end{aligned}\tag{7}$$

where $r_\ell = r_\ell(\beta, m, n)$ is a positive integer. Analogously

$$\begin{aligned}r_\ell(\beta, m, n)2 \sin\left(\frac{m}{n}\pi + \eta_{m,n}^+(\beta, \ell)\right) &= t \\ r_\ell(\beta, m, n)2 \sin\left(\frac{m}{n}\pi - \eta_{m,n}^-(\beta, \ell)\right) &= t\end{aligned}\tag{8}$$

Recall that $n \leq \lceil \frac{2\pi}{\Delta} \rceil$. Let $t > \frac{4\pi}{\Delta}$. Then $r_\ell(\beta, m, n) = K(\beta, m, n)n + \ell$ where $0 \leq \ell \leq n-1$. It easily follows from Lemmas 3 and 4 that $K(\beta_1, m, n) = K(\beta_2, m, n)$ for any $\beta_1, \beta_2 \in \text{proj}(\mathcal{N}_{\psi_{m,n,\ell}}^t)$, $0 \leq \ell < n$. Therefore, in what follows we will write $K(m, n)$ instead of $K(\beta, m, n)$.

From (7) and (8) we get after some tedious but elementary computations

$$\begin{aligned}1 - \frac{2}{K(m, n) - 1} &< \frac{\eta_{m,n}^+(\beta, j)}{\eta_{m,n}^+(\beta, i)} < 1 + \frac{3}{K(m, n) - 1} \\ 1 - \frac{2}{K(m, n) - 1} &\leq \frac{\eta_{m,n}^-(\beta, j)}{\eta_{m,n}^-(\beta, i)} < 1 + \frac{3}{K(m, n) - 1}\end{aligned}\tag{9}$$

for any pair of integers i, j , $0 \leq i \leq n-1$, $0 \leq j \leq n-1$. Another easy estimate gives that for any ℓ , $0 \leq \ell < n$

$$\begin{aligned}K(m, n) - 1 &< \frac{t}{2n \sin\left(\frac{m}{n}\pi + \eta_{m,n}^+(\beta, j)\right)} < K(m, n) + 1, \\ K(m, n) - 1 &< \frac{t}{2n \sin\left(\frac{m}{n}\pi - \eta_{m,n}^-(\beta, j)\right)} < K(m, n) + 1.\end{aligned}\tag{10}$$

Finally (2), (6), (7), (8), (9), and (10) imply the following estimate of $\mu(\mathcal{N}_t)$

$$\begin{aligned} & \frac{1}{4\pi} \sum_{\substack{m < n, (m,n)=1, \\ n < \lceil \frac{2\pi}{\Delta} \rceil}} \left(1 - \frac{2}{K(m,n) - 1} \right) \frac{n \left[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon) \right]}{t} \sin^2 \frac{m}{n} \pi \\ & < \mu(\mathcal{N}_t) < \frac{1}{4\pi} \sum_{\substack{m < n, (m,n)=1, \\ n < \lceil \frac{2\pi}{\Delta} \rceil}} \left(1 + \frac{3}{K(m,n) - 1} \right) \frac{n \left[g\left(\frac{2\pi}{n} - \theta' - \epsilon\right) + g(\theta' - \epsilon) \right]}{t} \sin^2 \frac{m}{n} \pi \end{aligned} \quad (11)$$

where $g(x) = \begin{cases} x^2, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$

Theorem 2

$$\begin{aligned} P_\infty(\theta, \Delta) &= \lim_{t \rightarrow \infty} t\mu(\mathcal{N}_t) = \frac{1}{8\pi} \sum_{n=1}^{\lceil 2\pi/\Delta \rceil} n(\phi(n) - \mu(n)) \\ &\quad \times \left[g\left(\frac{2\pi}{n} - \theta' - \Delta\right) + g(\theta' - \Delta) \right], \end{aligned} \quad (12)$$

where $\phi(n)$ is the Euler function and $\mu(n)$ is the Möbius function.

Proof. Clearly $\lim_{t \rightarrow \infty} K(m, n) = \infty$. We apply now the Ramanujan identity [5]

$$\sum_{\substack{m=0 \\ (m,n)=1}}^{n-1} e^{2\pi i m/n} = \mu(n) \quad (13)$$

Then

$$\begin{aligned} \sum_{\substack{m=0 \\ (m,n)=1}}^{n-1} \sin^2\left(\frac{\pi m}{n}\right) &= -\frac{1}{4} \sum_{\substack{m=0 \\ (m,n)=1}}^{n-1} (e^{\pi i m/n} - e^{-\pi i m/n})^2 = \\ &= -\frac{1}{4} \left(2\mu(n) - 2 \sum_{\substack{m=0 \\ (m,n)=1}}^{n-1} 1 \right) = \frac{\phi(n) - \mu(n)}{2}, \end{aligned}$$

which together with (11) imply (12).

5. The limit of small holes

The function $P_\infty(\theta, \Delta)$ is piecewise smooth with respect to each of θ and Δ . The sum in (12) is finite, which becomes infinite when $\Delta \rightarrow 0$.

We will study the limiting behavior of $P_\infty(\theta, \Delta)$ as $\Delta \rightarrow 0$. First, we change $\lceil 2\pi/\Delta \rceil$ onto ∞ in the upper limit of \sum in (12) just by formally adding to the sum terms identically equal zero for finite Δ .

Consider now the Mellin transform

$$\tilde{P}_\theta(s) = \int_0^\infty P_\infty(\theta, \Delta) \Delta^{s-1} d\Delta \quad (14)$$

The transform $\tilde{P}_\theta(s)$ exists if the integral $\int_0^\infty |P_\infty(\theta, \Delta)| \Delta^{k-1} d\Delta$ is bounded for some $k > 0$. It is certainly the case because $P_\infty(\theta, \Delta) = 0$ when $\Delta > \pi$, and therefore (14) converges for sufficiently large s .

Then the inverse transform

$$P_\infty(\theta, \Delta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Delta^{-s} \tilde{P}_\theta(s) ds \quad (15)$$

also exists if $c > k$, i.e., if c is greater than the real parts of all the poles of $\tilde{P}_\theta(s)$.

Write $\theta' = \frac{2\pi}{n} \left\{ \frac{n\theta}{2\pi} \right\}$, where $\{x\}$ is the fractional part of x . Then

$$\begin{aligned} \tilde{P}_\theta(s) &= \frac{1}{8\pi} \sum_{n=1}^{\infty} n(\phi(n) - \mu(n)) \left[\int_0^\infty \Delta^{s-1} \left(\frac{2\pi}{n} - \theta' - \Delta \right)^2 \right. \\ &\quad \left. + \int_0^\infty \Delta^{s-1} (\theta' - \Delta)^2 d\Delta \right] \\ &= \frac{1}{4\pi} \sum_{n=1}^{\infty} n(\phi(n) - \mu(n)) \left(\frac{\left(\frac{2\pi}{n} - \theta' \right)^{s+2} + \theta'^{s+2}}{s(s+1)(s+2)} \right) \\ &= \frac{1}{2} \frac{(2\pi)^{s+1}}{s(s+1)(s+2)} \sum_{n=1}^{\infty} \frac{\phi(n) - \mu(n)}{n^{s+1}} \left[\left(1 - \left\{ \frac{n\theta}{2\pi} \right\} \right)^{s+2} + \left\{ \frac{n\theta}{2\pi} \right\}^{s+2} \right] \end{aligned}$$

If s is sufficiently large then the convergence of the series (14) is uniform in Δ . Therefore we can interchange the sum and the integral in (14), and then integrate over Δ . The result is

$$\begin{aligned} P_\infty(\theta, \Delta) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds \Delta^{-s} (2\pi)^{s+1}}{2s(s+1)(s+2)} \sum_{n=1}^{\infty} \frac{\phi(n) - \mu(n)}{n^{s+1}} \\ &\quad \times \left[\left(1 - \left\{ \frac{n\theta}{2\pi} \right\} \right)^{s+2} + \left\{ \frac{n\theta}{2\pi} \right\}^{s+2} \right]. \end{aligned} \quad (16)$$

6. Rational angles between holes

In what follows we suppose that the angle θ between the holes H_1 and H_2 is a rational multiple of π , i.e., $\theta = 2\pi \frac{r}{q}$, $(r, q) = 1$. In particular, one gets a single hole case by letting $r = 0$, $q = 1$.

For any positive integer a consider the sum

$$\sum_{n \equiv a \pmod{q}} \frac{\phi(n) - \mu(n)}{n^{s+1}} \quad (17)$$

Transform now (17) by dividing all the terms through by the greatest common divisor $b = (a, q)$. Then we get

$$\sum_{n' \equiv a' \pmod{q'}} \frac{\phi(bn') - \mu(bn')}{(bn')^{s+1}} \quad (18)$$

where $n' = n/b$, $a' = a/b$, $q' = q/b$, where $(a', q') = 1$.

To make the paper self-contained we recall now some facts about the Dirichlet characters (see, e.g., [5] for more details). Dirichlet's characters to the modulus q are multiplicative functions $\chi(n)$ of an integer variable n which are periodic with period q . The conjugacy classes modulo q which are coprime to q form an abelian group under multiplication.

It is easy to see that the order of this group equals $\phi(q)$. Besides it is a finite abelian group. Therefore it has $\phi(q)$ irreducible representations $\chi(n)$ where $(n, q) = 1$. The characters $\chi(n)$ are in this case the complex roots of unity, i.e., $\chi(m)\chi(n) = \chi(mn)$. This definition is extended by setting $\chi(n) = 0$, if $(n, q) > 1$.

By the orthogonality relation [5]

$$\frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \chi(n) = \delta_{a,n} \quad (19)$$

where $\delta_{a,n} = 1$, if $a \equiv n \pmod{q}$, zero otherwise, and \bar{x} denotes a complex conjugate to a number x .

By inserting (19) into (18) we get

$$\sum_{n \equiv a \pmod{q}} \frac{\phi(n) - \mu(n)}{n^{s+1}} = \frac{1}{\phi(q')} \sum_{\chi} \bar{\chi}(a') \sum_{n'=1}^{\infty} \chi(n') \frac{\phi(bn') - \mu(bn')}{(bn')^{s+1}} \quad (20)$$

Let $n' = \prod_p p^{\alpha_p}$ be the decomposition of n' into prime factors. Then $\chi(n') = \prod_p \chi(p)^{\alpha_p}$. Furthermore

$$\mu(bn') = \begin{cases} \mu(b) \prod_p (-1)^{\alpha_p} & \text{if } bn' \text{ is square free} \\ 0 & \text{otherwise,} \end{cases} \quad (21)$$

$$\phi(bn') = \phi(b) \prod_{p|n', p|b} (1 - p^{-1}), \quad (22)$$

where $\alpha_p = 0$ if $p \nmid b$. Farther

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+1}} = \prod_p (1 - p^{-s-1}) = (\zeta(s+1))^{-1}, \quad (23)$$

where $\zeta(s)$ is the Riemann zeta function. Now by making use of the Möbius transform we get

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^{s+1}} = (\zeta(s+1))^{-1} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\zeta(s)}{\zeta(s+1)} \quad (24)$$

Therefore

$$\sum_n \frac{(\phi(n) - \mu(n))}{n^{s+1}} = \frac{\zeta(s) - 1}{\zeta(s+1)} \quad (25)$$

Analogously

$$\sum_n \frac{\chi(n)(\phi(n) - \mu(n))}{n^{s+1}} = \frac{L(s, \chi) - 1}{L(s+1, \chi)} \quad (26)$$

Finally we have

$$\begin{aligned} \sum_{n \equiv a \pmod{q}} \frac{\phi(n) - \mu(n)}{n^{s+1}} \\ = \frac{1}{b^{s+1}\phi(q')} \sum_{\chi} \frac{\bar{\chi}(a')(\phi(b)L(s, \chi) - \mu(b))}{L(s+1, \chi) \prod_{p|b} (1 - \chi(p)p^{-s-1})} \end{aligned} \quad (27)$$

where the characters are taken modulo q' and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad (28)$$

is the Dirichlet L function.

If $q' = 1$ then $L(s, \chi)$ reduces to the Riemann zeta function $\zeta(s)$. For each q' there is a trivial character $\chi(a')$ that assumes the value 1 for all a' coprime to q' . Therefore

$$L(s, 1) = \zeta(s) \prod_{p|q'} (1 - p^{-s}). \quad (29)$$

Let

$$\begin{aligned} \tilde{P}_{r/q}(s) &= \frac{(2\pi)^{s+1}}{2s(s+1)(s+2)} \sum_{a=1}^q \frac{\left(1 - \left\{\frac{ar}{q}\right\}\right)^{s+2} + \left\{\frac{ar}{q}\right\}^{s+2}}{b^{s+1}\phi(q')} \\ &\quad \times \sum_{\chi} \frac{\bar{\chi}(a')(\phi(b)L(s, \chi) - \mu(b))}{L(s+1, \chi) \prod_{p|b} (1 - \chi(p)p^{-s-1})}, \end{aligned} \quad (30)$$

where, as above, $b = (a, q)$, $a' = a/b$, $q' = q/b$ and the characters are taken mod q' . We note that odd characters (i.e. $\chi(-1) = -1$) and their L functions in the above expression cancel.

The function $\tilde{P}_{r/q}(s)$ has poles at $s = 0$, $s = -1$, $s = -2$, at zeros of $L(s+1, \chi)$ and at poles of $L(s, \chi)$. Dirichlet's function $L(s+1, \chi)$ with even χ has trivial zeros

q	$\tilde{P}(s)$
1	$\frac{(2\pi)^{s+1}(\zeta(s) - 1)}{2s(s+1)(s+2)\zeta(s+1)}$
2	$\frac{\pi^{s+1}\zeta(s)}{s(s+1)(s+2)\zeta(s+1)}$
3	$\frac{(2\pi/3)^{s+1}(3^s(7\zeta(s) + 2^{s+2}(\zeta(s) - 1) + 2) - \zeta(s)(2^{s+2} + 1))}{2s(s+1)(s+2)(3^{s+1} - 1)\zeta(s+1)}$
4	$\frac{(\pi/2)^{s+1}(2^s(13\zeta(s) + 3^{s+2}(\zeta(s) - 1) + 3) - \zeta(s)(3^{s+2} + 5))}{4s(s+1)(s+2)(2^{s+1} - 1)\zeta(s+1)}$
6	$\frac{[(\pi/3)^{s+1}(6^s + 8.12^s - 25.30^s + (1 - 3.2^s - 13.3^s - 8.4^s + 25.5^s + 27.6^s - 25.10^s + 8.12^s - 25.15^s + 25.30^s)\zeta(s))] \times [2s(s+1)(s+2)(2^{s+1} - 1)(3^{s+1} - 1)\zeta(s+1)]^{-1}}{1}$

Table 1. The function $\tilde{P}_{r/q}(s)$ (Eq. (30)) for $q = 1, 2, 3, 4, 6$ and $r = 1$.

at $s = -(2m + 1)$, where $m = 1, 2, \dots$ [6]. All other (nontrivial) zeros of $L(s + 1, \chi)$ have real part $\text{Re } s = -1/2$ assuming that the extended Riemann hypothesis that is concerned with the Dirichlet functions [5] is correct.

7. The simplest placements of two holes and the Riemann hypothesis

In this section we consider several specific values of q , when a number of characters does not exceed 2, i.e., $\phi(q) \leq 2$. There are thus five such values $q = 1, 2, 3, 4$ and 6. In all these cases the only even character is the trivial character, so the function $\tilde{P}_{r/q}(s)$ in (30) contains the Riemann zeta function and no other L functions.

Below there are two tables. The first table gives the exact expressions for the function $\tilde{P}_{1/q}(s)$ for $q = 1, 2, 3, 4, 6$ and $r = 1$. The second table contains the corresponding residues. (Except for the last lines for $q = 6$ these two tables were published in [1].)

We now list some properties of the Riemann zeta function that will be needed in what follows. Riemann proved that ζ -function satisfies to the following functional equation

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s) \quad (31)$$

which can also be written in the following (nonsymmetric) form

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos\left(\frac{\pi}{2}s\right) \Gamma(s) \zeta(s) \quad (32)$$

where $\Gamma(s)$ is the gamma function.

q	s			
	1	-1	-2	-3
1	2	$-\frac{13}{12}$	$\frac{3}{2\pi}$	$\frac{119}{5760\pi^2\zeta'(-2)}$
2	1	$-\frac{1}{6}$	0	$-\frac{1}{720\pi^2\zeta'(-2)}$
3	1	$-\frac{1}{4} - \frac{5\ln 2}{9\ln 3}$	$\frac{3}{4\pi}$	$\frac{49}{5120\pi^2\zeta'(-2)}$
4	1	$-\frac{1}{3} - \frac{11\ln 3}{16\ln 2}$	$\frac{3}{\pi}$	$\frac{109}{1620\pi^2\zeta'(-2)}$
6	1	$\frac{[5\ln 5(10\ln 3 - 7\ln 5) + \ln 2(55\ln 5 - 76\ln 3) + (10\ln 5 - 8\ln 2)(7\ln \Delta + 12\zeta'(-1))]}{\times [72\ln 2\ln 3]^{-1}}$	$-\frac{3}{2\pi}$	$-\frac{79}{6400\pi^2\zeta'(-2)}$

Table 2. Some residues of $\tilde{P}_{r/q}(s)\Delta^{-s}$ given in table 1 divided by the factor Δ^{-s} . The $\ln \Delta$ appears for $q = 6$ due to a double pole at $s = -1$. There are also poles for further negative odd s , and along the critical line $\Re s = -1/2$.

It is well known also that $\zeta(0) = -\frac{1}{2}$, $\zeta(-2m) = 0$,

$$\zeta(1 - 2m) = \frac{(-1)^m B_{2m}}{2m} \quad (33)$$

where $m = 1, 2, \dots$, and B_1, B_2, \dots are Bernoulli numbers. The following approximating formula holds for the Bernoulli numbers

$$B_{2m} \sim (-1)^{m-1} 4\sqrt{\pi m} \left(\frac{m}{\pi e}\right)^{2m}. \quad (34)$$

Another well known fact is that $\Gamma(s)$ has poles of order 1 at $s = -m$ for all integers $m > 0$ and

$$\text{Res}(\Gamma, -m) = \frac{(-1)^m}{m!} \quad (35)$$

where $\text{Res}(f, a)$ denotes the residue of the function $f(x)$ at the point $x = a$.

Lemma 5 *Let $q = 1, 2, 3, 4$ or 6, then*

$$\sum_j \text{Res}_{s=s_j}(\tilde{P}_{1/q}(s)\Delta^{-s}) < C\Delta |\ln \Delta|,$$

where $C > 0$ is a constant and the sum is taken over $s_j = -1, -2$ and over all trivial zeros of $\zeta(s+1)$, i.e., over all odd negative integers $m \leq -3$.

By combining (32), (33), (34), (35) with (30) and Stirling formula we obtain that

$$\left| \sum_j \text{Res}_{s=s_j}(\tilde{P}_{1/q}(s)) \right| < \sum_j |\text{Res}_{s=s_j}(\tilde{P}_{1/q}(s))| < \text{const}$$

and Lemma 5 follows.

Remark 1 The (extra) factor $|\ln \Delta|$ appears in the right hand side of the above estimate because of the double pole at $s = -1$ for $q = 6$. For $q = 1, 2, 3, 4$ this factor is not needed.

It is well known [6] that for $\tau \geq \tau_0 > 0$ uniformly in σ the following estimates hold

$$\zeta(\sigma + i\tau) = O \begin{cases} 1, & \sigma \geq 2 \\ \log \tau, & 1 \leq \sigma \leq 2 \\ \tau^{(1-\sigma)/2} \log \tau, & 0 \leq \sigma \leq 1 \\ \tau^{1/2-\sigma} \log \tau, & \sigma \leq 0. \end{cases} \quad (36)$$

We will consider now nontrivial zeros of the Riemann zeta function located at the critical strip $0 < \sigma < 1$. Let $N(t)$ denotes the number of zeros of $\zeta(s) = \zeta(\sigma + it)$ in the region $\{(\sigma, t) : 0 < \sigma < 1, 0 < t \leq T\}$ of the critical strip. Then [6]

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(\log T) \quad (37)$$

We now assume that the Riemann hypothesis (RH) is correct and use several of its well known [6] consequences.

Let $S(t)$ denote the multiplicity of the complex zero $S = \frac{1}{2} + it$ of $\zeta(s)$. Then on RH

$$S(t) = O\left(\frac{\log t}{\log \log t}\right) \quad (38)$$

We will construct an infinite sequence of contours C_n over which the integration of the function $\tilde{P}_{r/q}(s)$ (see (30)) will be performed in what follows. Each contour in this sequence will contain two vertical segments

$$I_n(k_0) = \{s = \sigma + i\tau : \sigma = k_0, -a_n \leq \tau \leq a_n\}$$

and $I'_n = \{s = \sigma + i\tau : \sigma = -b_n, -a_n \leq \tau \leq a_n\}$, where $b_n = 2n$, and two horizontal segments $I_n^+ = \{s = \sigma + i\tau : -b_n \leq \sigma \leq k_0, \tau = a_n\}$ and $I_n^- = \{s = \sigma + i\tau : -b_n \leq \sigma \leq k_0, \tau = -a_n\}$. We choose k_0 large enough to ensure uniform convergence of the Mellin transform (14) and assume from now on that $\Delta < 1$.

Lemma 6 *There exists an infinite sequence of contours C_n with $a_n \rightarrow \infty$ as $n \rightarrow \infty$ such that*

$$\lim_{n \rightarrow \infty} \int_{I'_n \cup I_n^+ \cup I_n^-} \tilde{P}_{r/q}(s) \Delta^{-s} ds = 0$$

for any entry in the Table 1 (i.e., for $q = 1, 2, 3, 4, 6$).

Proof. One gets from the Table 1 that $s = 1$ is the only pole of $\tilde{P}_{r/q}(s)$. Therefore we will assume that $k_0 > 1$. Besides, it is easy to see from the Table 1 that it is enough to consider instead of $\tilde{P}(s)$ the function

$$\hat{P}(s) = \frac{(2\pi)^{s+1}}{2q^{s+1}s(s+1)(s+2)} \frac{\zeta(s)}{\zeta(s+1)}.$$

We will start with the vertical segment I'_n . By making use of the functional equation (32) we get

$$\frac{\zeta(s)}{\zeta(s+1)} = -\frac{s}{2} \frac{\sin \frac{\pi}{2}s}{\sin \frac{(\pi+1)s}{2}} \frac{\zeta(1-s)}{\zeta(-s)} \quad (39)$$

Because of the relation $\overline{\zeta(\bar{s})} = \zeta(s)$, where \bar{s} denotes the complex conjugate to the complex number s , it is enough to consider $\int_{I_n^+} \hat{P}(s) \Delta^{-s} ds$. The estimates for $\int_{I_n^-} \hat{P}(s) \Delta^{-s} ds$ are quite analogous.

Clearly, the major problem with estimating of these integrals is caused by the zeros of $\zeta(s+1)$ in the denominator of $\hat{P}(s)$. Therefore we partition the horizontal segment I_n^+ into the union of three segments

$$\begin{aligned} I_{n,1}^+ &= \{s = \sigma + i\tau : -2n \leq \sigma \leq -1, \tau = a_n\}, \\ I_{n,2}^+ &= \{s = \sigma + i\tau : -1 < \sigma < 0, \tau = a_n\}, \quad \text{and} \\ I_{n,3}^+ &= \{s = \sigma + i\tau : 0 \leq \sigma \leq k_0, \tau = a_n\}. \end{aligned}$$

Observe that the lengths of $I_{n,2}^+$ and $I_{n,3}^+$ are constants. These sets are not exactly defined though because the values of a_n are not specified yet. We will make a choice of a_n now. To do that we assume the validity of the Riemann hypothesis. Then [6] each interval $(\tau, \tau+1)$ on the critical line $\sigma = \frac{1}{2}$ contains a value of τ such that

$$|\zeta(s)| > \exp \left(-A_1 \frac{\log \tau \log \log \log \tau}{\log \log \tau} \right) \quad (40)$$

where A_1 is an absolute constant.

We now choose a_n so that $0 < a_n \leq n+1$ and $\zeta(\frac{1}{2} + ia_n)$ satisfies (40). The next two estimates that we will use also hold on the assumption that the Riemann hypothesis is correct [6].

The first fact is that for any sufficiently small $\epsilon > 0$

$$\zeta(s) = O(\tau^\epsilon) \quad \text{and} \quad \frac{1}{\zeta(s)} = O(\tau^\epsilon) \quad (41)$$

if $\frac{1}{2} + \frac{1}{\log \log \tau} \leq \sigma$. Clearly (39) together with (41) gives that $\lim_{n \rightarrow \infty} \int_{I'_n} \hat{P}(s) \Delta^{-s} ds = 0$.

Another estimate works in the vicinity of the critical line $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log \log \tau}$ [6]

$$\log |\zeta(s)| > \frac{-A_2 \log \tau}{\log \log \tau} \log \left\{ \frac{2}{(\sigma - \frac{1}{2}) \log \log \tau} \right\}, \quad (42)$$

where A_2 is another absolute constant.

Now (40), (41), (42) and the last (fourth) estimate in (36) applied for $-1 \leq \sigma \leq 0$ imply that $\lim_{n \rightarrow \infty} \int_{I_{n,2}^+} \hat{P}(s) \Delta^{-s} ds = 0$.

In the estimates of the integrals of $\hat{P}(s)$ over $I_{n,1}^+$ and $I_{n,3}^+$ the term $s(s+1)(s+2)$ in the denominator ensures the needed results. Indeed, consider first $I_{n,3}^+$. Then (41) implies that $\int_{I_{n,3}^+} \hat{P}(s) \Delta^{-s} ds \xrightarrow[n \rightarrow 0]{} 0$. For $I_{n,1}^+$ we will again use the trick with the reflection (the functional equation). Then the relation (39) together with the estimates (41) ensures that on $I_{n,1}^+ \hat{P}(s)$ satisfies the inequality $|\hat{P}(s)| < |n|^\gamma$, where $0 < \gamma < 1$. The length of $I_{n,1}^+$ equals $2n$. Thus $\lim_{n \rightarrow \infty} \int_{I_{n,1}^+} \hat{P}(s) \Delta^{-s} ds = 0$ and Lemma 6 follows.

Lemma 7 *Assume that RH is correct. Let $q = 1, 2, 3, 4$ or 6 , $(r, q) = 1$, then for any $\alpha > 0$*

$$C_2 \Delta^{1/2} < \sum_j \operatorname{Res}_{s=\frac{1}{2}+i\tau_j} (\tilde{P}_{r/q}(s) \Delta^{-s}) < C_1 \Delta^{1/2-\alpha}, \quad (43)$$

where the sum is taken over all nontrivial zeros of $\zeta(s)$ and $C_1, C_2 > 0$ are some constants.

Proof. Observe, at first, that all expressions for $\tilde{P}_{r/q}(s)$ in the Table 1 have $\zeta(s+1)$ in denominators. Therefore all poles outside the real line correspond to zeros of $\zeta(s+1)$ and under RH are located on the line $s = -\frac{1}{2} + i\tau$, $-\infty < \tau < \infty$.

It is easy to see that residue at each zero of $\zeta(s+1)$ with multiplicity m results in the extra factor $(\ln \Delta)^{m-1}$ in the expression for the residue at this zero.

By making use of (38) we get for any fixed Δ and for sufficiently large τ

$$|\log \Delta|^{S(\tau)} < \tau^\delta \quad (44)$$

where $0 < \delta < 1/2$.

From (37), (38), (41) and (44) we have for sufficiently large T

$$\left| \sum_{j=1}^{\infty} \operatorname{Res}_{s=\frac{1}{2}+i\tau_j} (\tilde{P}_{1/q}(s)) \right| = \left| \sum_{n=0}^{\infty} \sum_{n \leq \tau_j \leq n+1} \operatorname{Res}_{s=\frac{1}{2}+i\tau_j} (\tilde{P}_{1/q}(s)) \right|$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} \left| \sum_{n \leq \tau_j < n+1} \text{Res}(\tilde{P}_{1/q}(s)) \right| \\
&\leq \sum_{n=0}^{\infty} \frac{(n+1)^{\delta+\epsilon}}{n^3} < \infty
\end{aligned} \tag{45}$$

Observe that convergence (or divergence) of the series in (42) is not influenced by the constant factor $\Delta^{1/2}$. Thus Lemma 7 follows.

From Lemmas 5, 6 and 7 follows

Theorem 3 *Consider a billiard in the unit circle with two holes $[0, \Delta]$ and $[2\pi\frac{r}{q}, 2\pi\frac{r}{q} + \Delta]$, where $q = 1, 2, 3, 4$ and 6, $0 < r < q$ are integers, $(r, q) = 1$. If $t > f(t)\Delta^{-1}$, where $f(t) > 0$ and $\lim_{t \rightarrow \infty} f(t) = \infty$, then*

$$P_{\infty}\left(\frac{r}{q}, \Delta\right) = \lim_{t \rightarrow \infty} \mu(\mathcal{N}_t) = \sum_k \text{Res}_{s=s_k} \tilde{P}_{r/q}(s) \Delta^{-s} \tag{46}$$

where summation is taken over all residues of the function $\tilde{P}_{r/q}(s)$.

Proof. Consider the sequence of contours C_n constructed in the proof of Lemma 6.

Then

$$\lim_{n \rightarrow \infty} \int_{C_n} \tilde{P}_{r/q}(s) \Delta^{-s} ds = \int_{k_0 - i\infty}^{k_0 + i\infty} \tilde{P}_{r/q}(s) \Delta^{-s} ds.$$

Farther applying the residue theorem and Lemmas 5 and 7 to the integral $\int_{C_n} \tilde{P}_{r/q}(s) \Delta^{-s} ds$ and letting $n \rightarrow \infty$ we obtain that $\int_{k_0 - i\infty}^{k_0 + i\infty} \tilde{P}_{r/q}(s) \Delta^{-s} ds = \sum_k \text{Res}_{s=s_k} \tilde{P}_{r/q}(s) \Delta^{-s} < \infty$. The relation (46) follows from (15) and (30).

Combining now Theorems 2 and 3 with Lemmas 5, 6 and 7 and with the first row in the Table 2 we obtain

Theorem 4 *Consider an open circular billiard with one hole (i.e., with two holes of the same length Δ placed on top of each other). Let $P_1(t, \Delta)$ denotes the probability that a particle will not escape till time t . Then, assuming that RH is correct, for any $\alpha > 0$*

$$\lim_{\Delta \rightarrow 0} \lim_{t \rightarrow \infty} \Delta^{\alpha-1/2} [tP_1(t, \Delta) - 2/\Delta] = 0 \tag{47}$$

The inverse statement is also true.

Theorem 4' *Consider the same open circular billiard as in Theorem 4. Then the validity of the relation (47) implies RH.*

Proof. Suppose that RH is not true. Then the Riemann zeta function $\zeta(s)$ has at least one zero s_0 in the critical strip which is outside the critical line, i.e., $s_0 = (\frac{1}{2} + \gamma + i\tau)$,

where $\gamma \neq 0$, $|\gamma| < \frac{1}{2}$. The functional equation (31) ensures that $\zeta(s_1) = 0$, where $s_1 = \frac{1}{2} - \gamma + it$. But either $\operatorname{Re} s_0 < \frac{1}{2}$ or $\operatorname{Re} s_1 < \frac{1}{2}$, which imply that instead of $\Delta^{\alpha-1/2}$ in (47) one must have a $\Delta^{\alpha-\min(\operatorname{Re} s_0, \operatorname{Re} s_1)}$ unless the sum of residues of $\tilde{P}_{1/q}(s)\Delta^{-s}$ over all zeros in the critical strip with the real parts less than $\frac{1}{2}$ is identically (for all $0 < \Delta \leq \Delta_0$) equal to zero. It is easy to see that it cannot occur. Thus we come to the contradiction which proves Theorem 4'.

Consider now circular billiard with two opposite (symmetric with respect to the center) holes with lengths Δ . Denote by $P_2(t, \Delta)$ probability that the billiard particle will not escape from this circle till time t .

From the first two rows in the Tables 1 and 2 we have that the relation (47) is equivalent to the statement that

$$\lim_{\Delta \rightarrow 0} \lim_{t \rightarrow \infty} \Delta^{\alpha-1/2} [tP_1(t, \Delta) - 2tP_2(t, \Delta)] = 0 \quad (48)$$

for any $\alpha > 0$. Thus one can formulate the analogs of the Theorems 4 and 4' by substituting (48) instead of (47). Therefore RH is equivalent to (48) which relates asymptotics of probabilities to escape in a circular billiard with one and with two symmetric holes.

Certainly one can use another rows in Tables 1 and 2 to formulate statements equivalent to RH in open billiards with two holes places under the angles $\frac{2\pi}{3}$, $\frac{\pi}{2}$ and $\frac{\pi}{3}$.

Moreover, in fact the analogous statements hold for open circular billiards with any number of q holes with lengths Δ which are equally spaced over the circle on the angle $\frac{2\pi}{q}$.

Theorem 5 *Consider an open circular billiard with $q \geq 2$ holes of the same length Δ with the centers placed at the vertices of a right convex q -angle. Let $P_q(t, \Delta)$ denotes the probability that the particle will not escape through this system of q (different) holes and $P_1^{(q)}(t, \Delta)$ denotes the probability that the particle will not escape till time t in case when all these holes are placed on top of each other. Then, assuming that RH is correct, for any $\alpha > 0$*

$$\lim_{\Delta \rightarrow 0} \lim_{t \rightarrow \infty} \Delta^{\alpha-\frac{1}{2}} t [P_1^{(q)}(t, \Delta) - qP_q(t, \Delta)] = 0 \quad (49)$$

The proof of Theorem 5 is completely analogous to the proof of Theorem 4. Theorem 1 remains valid for this system. The formula in Theorem 2 becomes

$$P_q(\Delta) = \lim_{t \rightarrow \infty} t\mu(\mathcal{N}_t(q)) = \frac{1}{8\pi} \sum_{n \geq 1} n(\phi(n) - \mu(n))\tilde{q}g\left(\frac{2\pi}{n\tilde{q}} - \Delta\right)$$

where $\tilde{q} = q/gcf(n, q)$ and the sum is over n for which the argument of the g function is positive. The sum is then written in terms of $\tilde{n} = n/gcf(n, q)$ with the result

$$P_q(\Delta) = \frac{1}{8\pi} \sum_{\tilde{n}=1}^{[2\pi/\Delta]} \tilde{n}\phi(\tilde{n})q^2g\left(\frac{2\pi}{\tilde{n}} - \Delta\right)$$

The Mellin transform of this is then

$$\tilde{P}_q(s) = \frac{(2\pi)^s \zeta(s)}{q^s s(s+1)(s+2)\zeta(s+1)} \quad (50)$$

for which the case $q = 2$ has already been given in Table 1.

The proof of the next statement is completely analogous to the one of Theorem 4'.

Theorem 5' *Consider the same open circular billiard with q uniformly placed holes as in Theorem 4. Then the relation (49) implies RH.*

8. Concluding Remarks

It is quite likely that the results of this paper can be readily generalized for Dirichlet L functions. Indeed, the main formula (30) for two holes escape with arbitrary rational angles between holes explicitly involves all $L(s, \chi)$ with even nontrivial characters χ . We conjecture that the corresponding $\Delta^{1/2}$ asymptotics for the two holes escape is equivalent to the extended RH for even characters. We reserve the term generalized RH for more general L -functions over number fields, elliptic curves, etc. It seems very interesting though whether the generalized RH is equivalent to a particular asymptotics of the escape in some specific classes of open dynamical systems. The most natural candidates for these systems are geodesic and contact flows on manifolds.

Another natural further problem is to compute the second order asymptotics of the escape from the open circular billiard through two holes placed under irrational (mod π) angles. The leading order (in Δ) behavior in this case remains the same though [1].

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